Iterative Methods — 4/19/06

Note Title 4/19/2006

$$\underset{\sim}{x}^{k+1} = B\underset{\sim}{x}^{k} + M^{-1}\underset{\sim}{b}$$

$$A = M - N \qquad \text{"splitting"}$$

"pre-conditioning"

$$\underset{\sim}{y}^{(k)} = \sum_{l=0}^{k} a_{kl}\, \underset{\sim}{x}^{(l)}$$

$$\sum_{l=0}^{k} a_{kl} = 1$$

$$-\hat{x} = -\sum_{l=0}^{k} a_{kl}\, x$$

$$\underset{\sim}{y}^{(k)} - \underset{\sim}{x} = \sum_{l=0}^{k} a_{kl} \left(\underset{\sim}{x}^{l} - \underset{\sim}{x} \right)$$

$$\underset{\sim}{x}^{l} - \underset{\sim}{x} = B \left(\underset{\sim}{x}^{l-1} - \underset{\sim}{x} \right) = B^{l} \left(\underset{\sim}{x}^{0} - \underset{\sim}{x} \right)$$

$$\underset{\sim}{y}^{k} - \underset{\sim}{x} = \sum_{l=0}^{k} a_{kl} \, B^{l} \left(\underset{\sim}{x}^{0} - \underset{\sim}{x} \right)$$

$$= P_k (B) \left(\underset{\sim}{x}^{0} - \underset{\sim}{x} \right)$$

$$P_k(\lambda) = \sum_{l=0}^{k} a_{kl} \lambda^{l}, \quad P_k(1) = 1$$

$$\left(y^k - \underset{\sim}{x}\right) = P_k(B)\left(\underset{\sim}{x}^0 - \underset{\sim}{x}\right)$$

$$\left\| y^k - \underset{\sim}{x} \right\| = \left\| P_k(B)\left(\underset{\sim}{x}^0 - \underset{\sim}{x}\right) \right\|$$

$$\leq \left\| P_k(B) \right\| \left\| \underset{\sim}{x}^0 - \underset{\sim}{x} \right\|$$

$$\left\| P_k(B) \right\|_2 = \max_{\{\lambda_i\}} \left| P_k(\lambda) \right|$$

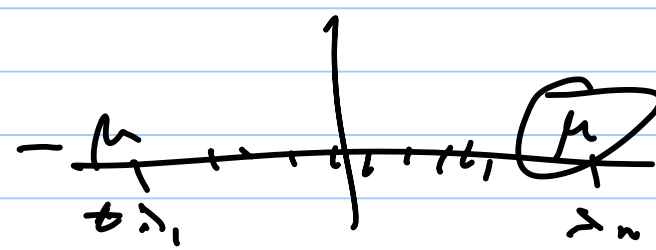
Objective $\min_{\{\lambda_i\}} \max \left| P_k(\lambda) \right|$ $\qquad$ $P_k(1) = 1$

$$\max_{\{\lambda_i\}} |P_k(\lambda_i)| \le \max_{\lambda_1 \le \lambda \le \lambda_n} |P_k(\lambda)|$$

$$T_0(z) = 1, \quad T_1(z) = z$$

$$T_{k+1}(z) = 2z\,T_k(z) - T_{k-1}(z)$$

$|z| \le 1$ $\qquad$ $\lambda_1 = -\mu, \ \lambda_n = \mu$



$$P_k(\lambda) = \frac{T_k\left(\frac{\lambda}{\mu}\right)}{T_k\left(\frac{1}{\mu}\right)} \quad \left\{ \begin{array}{l} \max\limits_{-\mu \le \lambda \le \mu} \left| P_k(\lambda) \right| = \min. \\ \underline{P_k(1) = 1} \end{array} \right.$$

$$T_{k+1}\left(\frac{1}{\mu}\right) \cdot P_{k+1}(z) = \frac{2z}{\mu} T_k\left(\frac{1}{\mu}\right) P_k(z) - T_{k-1}\left(\frac{1}{\mu}\right) P_{k-1}(z)$$

$$\underset{\sim}{y}^{k+1} - \underset{\sim}{x} = P_k(B)\left(\underset{\sim}{x}^0 - \underset{\sim}{x}\right) = \frac{2\, T_k\left(\frac{1}{\mu}\right)}{\mu\, T_{k+1}\left(\frac{1}{\mu}\right)} B \left(\underset{\sim}{y}^k - \underset{\sim}{x}\right)$$

$$- \frac{T_{k-1}\left(\frac{1}{\mu}\right)}{T_{k+1}\left(\frac{1}{\mu}\right)} \left(\underset{\sim}{y}^{k-1} - \underset{\sim}{x}\right)$$

$$\Big| \underset{\sim}{y}^{(k)} = \sum_{i=1}^{k} a_{ki}\, \underset{\sim}{x}^{i} \Big)$$

$$\underset{\sim}{x} = B\underset{\sim}{x} + M^{-1}\underset{\sim}{b} \qquad \left(\underset{\sim}{x} = B\underset{\sim}{x} + M^{-1}\underset{\sim}{b}\right)$$

After some manipulations

$$\underset{\sim}{y}^{k+1} = \omega_{k+1}\left(B\underset{\sim}{y}^{k} + M^{-1}\underset{\sim}{b} - \underset{\sim}{y}^{k-1}\right) + \underset{\sim}{y}^{k-1}$$

$$\omega_{k+1} = \frac{2T_k\left(\frac{1}{\mu}\right)}{\mu\, T_{k+1}\left(\frac{1}{\mu}\right)} = \frac{1}{1 - \frac{\mu^2}{4}\omega_k}$$

$$A = M - N, \quad B = I - M^{-1}A$$

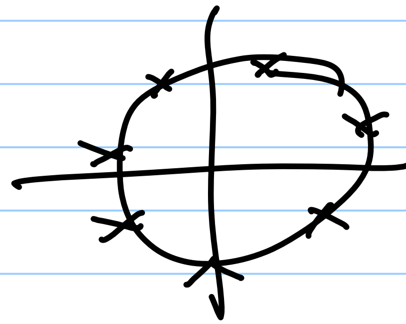
$$\underset{\sim}{y}^{k+1} = \omega_{k+1}\left(\underset{\sim}{z}^k + \underset{\sim}{y}^k - \underset{\sim}{y}^{k-1}\right) + \underset{\sim}{y}^{k-1}.$$

$$M\underset{\sim}{z}^k = \underset{\sim}{r}^k = (\underset{\sim}{b} - A\underset{\sim}{y}^k)$$

Original process: $\underline{\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^k + \underset{\sim}{z}^k}$

Chebyshev semi-iterative method

(S.O.R.



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$P_k(z) = z^k.$

$M = \underline{I - \alpha A}, \quad \alpha = \dfrac{2}{\lambda_1 + \lambda_n}$

$\lambda(M) = 1 - \dfrac{2}{\lambda_1 + \lambda_n} \cdot \lambda_1 \qquad \lambda(M) \le \dfrac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}$

$$\frac{\|y^{(k)} - x\|}{\|x^0 - x\|} \leq \frac{\text{const.}}{T_k\left(\frac{1}{\mu}\right)}$$

$$T_k\left(\frac{1}{\mu}\right) = \cosh\left(k \cosh^{-1} \frac{1}{\mu}\right)$$

$$1 < \omega_k < 2$$

$$\omega_{k+1} = \frac{1}{1 - \frac{\mu^2}{4}\omega_k} \qquad \omega_k \downarrow \hat{\omega}$$

$$\hat{\omega} = \frac{1}{1 - \frac{\mu^2}{4}\hat{\omega}} \qquad 1 < \omega < 2$$

$$\hat{\omega} - \frac{\mu^2}{4}\hat{\omega}^2 = 1, \quad \frac{\mu^2}{4}\hat{\omega}^2 - \hat{\omega} + 1 = 0$$

$$1 < \hat{\omega} = \frac{2}{1 + \sqrt{1-\mu^2}}$$ (opt. S.O.R parameter)

$$A = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix} \qquad M = I, \quad N = \begin{pmatrix} 0 & -F \\ -F^T & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -F \\ -F^T & 0 \end{pmatrix}$$

$$A\tilde{y} = \begin{pmatrix} \tilde{y}_1 + F\tilde{y}_2 \\ F^T\tilde{y}_1 + \tilde{y}_2 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{y}_1^{k+1} \\ \tilde{y}_2^{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{y}_1^{k-1} \\ \tilde{y}_2^{k-1} \end{pmatrix} + \omega_{k+1} \begin{pmatrix} \tilde{b}_1 - F\tilde{y}_2^{k}, & - \tilde{y}_1^{k-1} \\ \tilde{b}_2 - F^T\tilde{y}_1^{k}, & - \tilde{y}_2^{k-1} \end{pmatrix}$$

$$\begin{array}{cccc} 2k+1 & 2k-1 & & 2k \\ \begin{pmatrix} \times \\ \end{pmatrix}, & \begin{pmatrix} \times \\ \end{pmatrix} & & \begin{pmatrix} \\ \times \end{pmatrix} \end{array}$$

$\tilde{y}_1^{2k+1}$; $\tilde{y}_1^{2k-1}$, $\tilde{y}_2^{2k}$

$\tilde{y}_2^{2k}$; $\tilde{y}_1^{2k-1}$, $\tilde{y}_2^{(2k-2)}$

$$\begin{array}{ccc} \begin{pmatrix} \\ \times \end{pmatrix} & \begin{pmatrix} \times \\ \bullet \end{pmatrix} & \begin{pmatrix} \\ \times \end{pmatrix} \\ 2k & 2k-1 & 2k-2 \end{array}$$

$\tilde{y}_1^{2k+1}$, $\tilde{y}_2^{2k}$

S.O.R $\equiv$ Chebyshev (skipping) of
a sequence of ω_n's.

What value μ?

$$p_{n+1} = (x - \alpha_{n+1})\, p_n - \beta_n^2\, p_{n-1}$$

$$\int p_k\, p_l\, w(x)\, dx = 0 \qquad k \neq l$$

$$\underset{\sim}{x}^{l+1} = \underset{\sim}{x}^{l-1} + \omega_{l+1}\left(\alpha_l\, \underset{\sim}{z}^{l} + \underset{\sim}{x}^{l} - \underset{\sim}{x}^{l-1}\right)$$

$$M \underset{\sim}{z}^{l} = \underset{\sim}{r}^{l} \qquad M = M^T,\ \text{p.d.}$$

$$(\underset{\sim}{z}^{l}, M \underset{\sim}{z}^{l+1}) = 0, \quad (\underset{\sim}{z}^{l-1}, M \underset{\sim}{z}^{l+1}) = 0$$

$$(\underset{\sim}{z}^{k}, M\underset{\sim}{z}^{(\ell+1)}) = 0 \qquad k < \ell - 1$$

$$(\underset{\sim}{z}^{j}, M\underset{\sim}{z}^{k}) = 0 \qquad j \neq k,$$

~~Assumption~~ $$(\underset{\sim}{z}^{j}, M\underset{\sim}{z}^{\ell}) = 0 \qquad j = 0, 1, \dots, \ell$$

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^{k-1} + \omega_{k+1}\left(\alpha_k \underset{\sim}{z}^{k} + \underset{\sim}{x}^{k} - \underset{\sim}{x}^{k-1}\right)$$

$$\underset{\sim}{r}^{k} = \underset{\sim}{b} - A\underset{\sim}{x}^{k} = A(\underset{\sim}{x} - \underset{\sim}{x}^{k}), \qquad M\underset{\sim}{z}^{k} = r^{k}$$

$$(\underset{\sim}{b} - A\underset{\sim}{x}^{k+1}) = \underset{\sim}{b} - A\underset{\sim}{x}^{k-1} - \omega_{k+1}\left(\alpha_k A\underset{\sim}{z}_k + A\underset{\sim}{x}^{k} - \underset{\sim}{b} + \underset{\sim}{b} - A\underset{\sim}{x}^{k-1}\right)$$

$$r^{k+1} = r^{k-1} - \omega_{k+1}\left(\alpha_k A z^k - r^k + r^{k-1}\right)$$

$$M z^k = r^k$$

$$M z^{k+1} = M z^{k-1} - \omega_{k+1}\left(\alpha_k A z^k - M z^k + M z^{k-1}\right)$$

$$z^{k+1} = z^{k-1} - \omega_{k+1}\left(\alpha_k M^{-1} A z_k - z^k + z^{k-1}\right)$$

$$(z^j, M z^{k+1}) = (z^j, M z^{k-1}) - \omega_{k+1}\Big(\alpha_k (z^j, A z^k) - (z^j, M z^k) + (z^j, M z^{k-1})\Big)$$

$$\underline{(z^k, M z^{k+1})} = (z^k, M z^{k-1}) - \omega_{k+1}\Big(\alpha_k (z^k, A z^k) - (z^k, M z^k) + (z^k, M z^{k-1})\Big)$$

$$0 = 0 - \omega_{k+1}\left(\alpha_k (\underset{\sim}{z}^k, A\underset{\sim}{z}^k) + (\underset{\sim}{z}^k, M\underset{\sim}{z}^k)\right)$$

$$\alpha_k = \left[\frac{(\underset{\sim}{z}^k, A\,\underset{\sim}{z}^k)}{(\underset{\sim}{z}^k, M\,\underset{\sim}{z}^k)}\right]^{-1} \geq 0.$$

$$M\underset{\sim}{z}^k = \underset{\sim}{r}^k$$