CME 324: ITERATIVE METHODS
SPRING 2005/06
ASSIGNMENT 1

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1. Consider an $n \times n$ tridiagonal matrix of the form

$$T_\alpha = \begin{bmatrix} \alpha & -1 & & & & \\ -1 & \alpha & -1 & & & \\ & -1 & \alpha & -1 & & \\ & & -1 & \alpha & -1 & \\ & & & -1 & \alpha & -1 \\ & & & & -1 & \alpha \end{bmatrix},$$

where α is a real parameter.

(a) Verify that the eigenvalues of T_α are given by

$$\lambda_j = \alpha - 2\cos(j\theta), \qquad j = 1, \dots, n,$$

where

$$\theta = \frac{\pi}{n+1}$$

and that an eigenvector associated with each λ_j is

$$\mathbf{q}_j = [\sin(j\theta), \sin(2j\theta), \dots, \sin(nj\theta)]^\top.$$

Under what condition on α does this matrix become positive definite?

(b) Now take $\alpha = 2$.

(i) Will the Jacobi iteration converge for this matrix? If so, what will its convergence factor be?

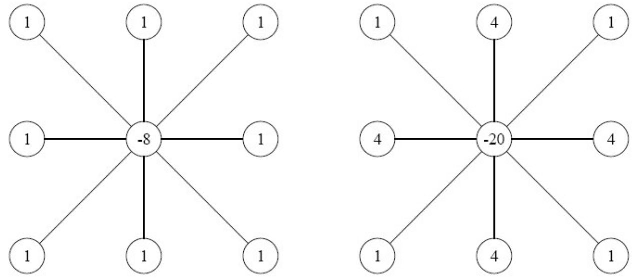
(ii) Will the Gauss-Seidel iteration converge for this matrix? If so, what will its convergence factor be?

(iii) For which values of ω will the SOR iteration converge?

2. Prove that the iteration matrix G_ω of SSOR can be expressed as

$$G_\omega = I - \omega(2-\omega)(D-\omega F)^{-1} D (D-\omega E)^{-1} A.$$

3. We are interested in solving Poisson's equation on a rectangle with $h = 1/(n+1)$. We want to use a nine-point formula; i.e.



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This assignment is due in class on Monday, May 1.

Thus,

$$A = \begin{bmatrix} T & B & & & \\ B & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & B \\ & & & B & T \end{bmatrix}.$$

where the matrices T and B are tridiagonal.

(a) Write down the matrices T and B.

(b) Give the eigenvalues and eigenvectors of T and B.

(c) Show that $TB = BT$.

(d) Find the eigenvalues and eigenvectors of A. (*Hint*: First, diagonalize T and B and then reorder the rows and columns so that the matrix is block diagonal.)

(e) Consider the block Jacobi algorithm:

$$T\mathbf{x}_j^{(k+1)} = \mathbf{b}_j - B\mathbf{x}_{j-1}^{(k)} - B\mathbf{x}_{j+1}^{(k)}.$$

Give the spectral radius of $M^{-1}N$.

(f) Determine the optimal $\hat{\omega}$ for the SOR method.

(g) Consider the differential equation

$$\begin{aligned} -u_{xx} - u_{yy} &= -12x^2 - 24, \qquad 0 < x < 1, \quad 0 < y < 1, \\ u(0, y) &= 12y^2, \\ u(1, y) &= 1 + 12y^2, \\ u(x, 0) &= x^4, \\ u(x, 1) &= x^4 + 12. \end{aligned}$$

The solution is $u(x, y) = x^4 + 12y^2$. Solve the differential equation using SOR for $h = 1/50$. Use both the natural ordering and the red/black ordering. As an initial vector, use $\mathbf{x} = \mathbf{0}$. Use the optimal $\hat{\omega}$ and $\omega = 1.0$ and see how the number of iterations differ.